

# An Arctangent Law

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## Abstract

Let  $M_r$  be the maximum value of an one-dimensional Brownian motion on the (time) interval  $[0, r]$ . We derive an explicit formula for the distribution of the time required (after  $r$ ) for the Brownian motion to exceed  $M_r$ .

**Keywords.** Brownian motion; maximum value on an interval.

**2010 AMS Mathematics Classification.** 60J65.

## 1 Introduction—The Theorem

Let  $B = \{B_t\}_{t \geq 0}$  be a standard Brownian motion in  $\mathbb{R}^1$ . The distribution of  $B_0$  can be arbitrary. We set

$$M_r := \max_{0 \leq t \leq r} B_t,$$

where  $r > 0$  is a fixed time. The object of this brief note is the calculation of the distribution of the random time

$$S := \inf\{t \geq r : B_t = M_r\} - r = \inf\{t \geq r : B_t > M_r\} - r \quad (1.1)$$

(obviously,  $S + r$  is a stopping time for  $B$ ).

**Theorem.** *The distribution function of the random variable  $S$  defined in (1.1) is*

$$F_S(s) := P\{S \leq s\} = \frac{2}{\pi} \arctan \left( \sqrt{\frac{s}{r}} \right), \quad s \geq 0. \quad (1.2)$$

It is remarkable that  $F_S(s)$  is an elementary function.

## 2 A Lemma

Suppose that  $X > 0$  is a random variable and  $W = \{W_t\}_{t \geq 0}$  is an one-dimensional Brownian motion with  $W_0 = 0$ . The passage time of  $W$  to the level  $X$  is

$$T_X := \inf\{t \geq 0 : W_t = X\}. \quad (2.1)$$

We remind the reader that if  $X$  is not random, say  $X = x > 0$ , then the reflection principle (see, e.g., [1]) yields

$$P\{T_x \leq t\} = 2P\{W_t \geq x\} = \sqrt{\frac{2}{\pi}} \int_{x/\sqrt{t}}^{\infty} e^{-\xi^2/2} d\xi, \quad t > 0. \quad (2.2)$$

**Lemma.** *If  $X$  and  $W$  are independent, then the density of  $T_X$  is*

$$f_{T_X}(t) = \frac{1}{\sqrt{2\pi t^3}} E\left[X e^{-X^2/2t}\right], \quad t > 0. \quad (2.3)$$

*Proof.* For  $t > 0$  we have

$$\begin{aligned} P\{T_X \leq t\} &= \int_0^{\infty} P\{T_X \leq t, X \in dx\} = \int_0^{\infty} P\{T_X \leq t \mid X = x\} dF_X(x) \\ &= \int_0^{\infty} P\{T_x \leq t \mid X = x\} dF_X(x) = \int_0^{\infty} P\{T_x \leq t\} dF_X(x), \end{aligned}$$

where the last equality follows from the independence of  $X$  and  $W$ . Thus, by invoking (2.2)

$$P\{T_X \leq t\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left[ \int_{x/\sqrt{t}}^{\infty} e^{-\xi^2/2} d\xi \right] dF_X(x), \quad (2.4)$$

from which (2.3) follows by differentiation with respect to  $t$  (the passing of  $d/dt$  inside the integral with respect to  $x$  is justified by the fact that the quantity  $x \exp(-x^2/2t)$  is bounded in  $x$ , for any  $t > 0$ ).  $\blacksquare$

Before closing this short section it maybe worth noticing that, by applying integration by parts (or Tonelli) in the integral of (2.4) one obtains the formula

$$P\{T_X \leq t\} = \sqrt{\frac{2}{\pi t}} \int_0^{\infty} e^{-x^2/2t} F_X(x) dx, \quad t > 0,$$

where  $F_X(x)$  is the distribution function of  $X$ .

### 3 Proof of the Theorem

As before, let  $r$  be a given positive number. We set

$$X = M_r - B_r \quad \text{and} \quad W_t = B_{t+r} - B_r, \quad t \geq 0. \quad (3.1)$$

Clearly,  $X > 0$  a.s.,  $W = \{W_t\}_{t \geq 0}$  is a Brownian motion with  $W_0 = 0$  and, also,  $W$  and  $X$  are independent. Furthermore, as it was observed by P. Lévy (see, e.g., [1], p. 97),  $X$  and  $|B_r - B_0|$  have the same law and, therefore, the density of  $X$  is

$$f_X(x) = \sqrt{\frac{2}{\pi r}} e^{-x^2/2r}, \quad x > 0. \quad (3.2)$$

We are now ready to complete the proof of our result.

*Proof of the theorem.* Observe that, in view of (2.1) the random time  $S$  of (1.1) can be expressed as

$$S = T_X,$$

where  $X$  is given by (3.1). Thus, we can use (2.3) to obtain the density of  $S$

$$f_S(s) = \frac{1}{\sqrt{2\pi s^3}} \int_0^\infty x e^{-x^2/2s} f_X(x) dx, \quad s > 0,$$

where  $f_X(x)$  is given by (3.2). It follows that

$$f_S(s) = \frac{1}{\pi \sqrt{rs^3}} \int_0^\infty x e^{-x^2/2s} e^{-x^2/2r} dx = \frac{\sqrt{r}}{\pi} \cdot \frac{1}{(s+r)\sqrt{s}}, \quad s > 0,$$

from which (1.2) follows immediately by integration with respect to  $s$ . ■

**Remark.** A remarkable consequence of the theorem is the following: For given  $r_1, r_2$ , with  $0 \leq r_1 < r_2$ , we set

$$M_{[r_1, r_2]} := \max_{r_1 \leq t \leq r_2} B_t$$

and

$$S_{[r_1, r_2]} := \inf \{t \geq r_2 : B_t = M_{[r_1, r_2]}\} - r_2.$$

Then,

$$P \{S_{[r_1, r_2]} \leq s\} = \frac{2}{\pi} \arctan \left( \sqrt{\frac{s}{r_2 - r_1}} \right), \quad s \geq 0.$$

### References

- [1] I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Second Edition, Springer, New York, 1991.